(Non)convergence of the finite element method on very general meshes

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The classical theory of finite element method (FEM) convergence states that piecewise linear finite elements in 2D converge to the exact solution with reasonable regularity, provided the triangles in the mesh satisfy the maximum angle condition [1]. Under this condition the triangles cannot degenerate in the sense that their maximum angles must be uniformly bounded away from 180 degrees. Babuška and Aziz tried (and failed) to prove that this sufficient condition is also necessary, however they provided a first counterexample to finite element convergence, [1]. As was later shown by Hannukainen et al. by a simple argument in [2], FEM triangulations can contain arbitrarily many arbitrarily degenerate triangles spread throughout the domain and still have optimal convergence rates.

In this talk we will review these results and present an overview of the paper [3], where a finer analysis of necessary and sufficient conditions for FEM convergence of general orders is presented. The key in the analysis is the investigation of locking phenomena which occur when degenerating elements form certain large enough structures. Understanding where exactly the limits of the FEM lie is important when trying to construct robust schemes based on extremely deformed meshes (X-MESH). Gaining deeper insight into these problems could perhaps lead to modifications of the FEM that would converge on more general meshes, or perhaps even on any possible mesh - a possibility that we will discuss.

[1] Babuška, I.; Aziz, A.K.: On the angle condition in the finite element method. SIAM J. Numer. Anal. 13(2), 214 - 226 (1976).

[2] Hannukainen, A.; Korotov, S.; Křížek, M.: The maximum angle condition is not necessary for convergence of the finite element method. Numer. Math. 120 (1), 78 - 88 (2012).

[3] Kučera, V.: On necessary and sufficient conditions for finite element convergence. Preprint: https://arxiv.org/abs/1601.02942.