Porous Medium Equation resolution with the eXtreme Mesh deformation approach (X-Mesh)

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The Porous Medium Equation appears in a wide range of applications, including: modelization of gas flow through a porous medium, heat propagation by radiation in plasmas at very high temperatures, groundwater flow and population dynamics. This equation can be expressed as:

$$\begin{cases} \partial_t u = \nu \nabla^2 u^m, \text{ with } u \geq 0,\ \nu > 0 \text{ and } m > 1. \\ [\nu] = m^2.s^{-1} \\ u \text{ dimensionless} \end{cases}$$

A remarkable characteristic of the Porous Medium Equation is the emergence of a time-dependent interface when the initial solution is zero in some domain of the space. This interface dynamically separates the phase regions, where the solution is strictly positive, from the empty regions, where the solution remains zero. Depending on the value of m, the solution's spatial gradient for this one-phase problem can exhibit discontinuities and even be infinite at the interface. The existence of this time-dependent interface, in combination with the positivity constraint and the presence of discontinuous (and potentially infinite) spatial gradient, poses significant challenges for solving such problems using the finite element method.

In this work, we present a method to solve the Porous Medium Equation without any regularization for any $m > 1$, utilizing the eXtreme Mesh deformation approach (X-Mesh). The X-Mesh approach, initially developed for two-phase problems [1], enables to follow sharp interfaces and their possible topology changes without requiring remeshing or altering mesh topology. To adapt this approach to the distinctive behavior of the Porous Medium Equation, we introduce three stages that diverge from the original X-Mesh formulation.

First, a different node selection strategy to capture the interface accurately is proposed. Second, we employ an alternative node positioning technique based on the weak formulation residual. Finally, we propose a treatment for the time-dependent mesh tailored specifically to one-phase problems, using the Arbitrary Lagrangian-Eulerian method.

By incorporating these modifications, our approach provides an efficient and accurate method for computing solutions for the Porous Medium Equation, overcoming the challenges posed by the time-dependent interface and positivity constraint.

References

[1] N. Moes, J.-F. Remacle, J. Lambrechts, B. Le, and N. Chevaugeon. The eXtreme Mesh deformation approach (X-MESH) for the Stefan phase change model. *Journal of Computational Physics*, page 111878, 2023.